Periodic solutions of Schrodinger equation in Hilbert space.

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Necessary and sufficient conditions for existence of boundary value problem of Schrodinger equation are obtained in linear and nonlinear cases. Periodic analytical solutions are represented using generalized Green's operator.

Auxilary result(Linear case).

Statement of the problem. Consider the next boundary value problem for Shrodinger equation

$$\frac{d\varphi(t)}{dt} = -iH_0\varphi(t) + f(t), t \in [0; w] \tag{1}$$

$$\varphi(0) - \varphi(w) = \alpha \in D \tag{2}$$

in a Hilbert space H_T, where, for each $t \in [0; w]$, the unbounded operator H_0 has the form [1]

$$H_0 = i\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix},$$

for simplicity. In more general case operator H_0 has the next form

$$H_0 = iJ\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = i\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}J, \quad J = J^* = J^{-1},$$

where T is strongly positive self-adjoint operator in the Hilbert space H. Since operator T^2 is closed, then domain $D(T)$ of operator T is Hilbert space with scalar product (Tu, Tu). The space $H_T = H \oplus H$ and operator H_0 is self-adjoint on domain $D = D(T) \oplus D(T)$ with product

$$(< u, v >, < u, v >)_{H^T} = (Tu, Tu)_H + (Tv, Tv)_H$$

and infinitesemal generator of strongly continuous evolution semigroup

$$U(t) := U(t,0) = \begin{pmatrix} cos\ tT & sin\ tT \\ -sin\ tT & cos\ tT \end{pmatrix}, \quad U^n(t) = \begin{pmatrix} cos\ ntT^{\frac{1}{2}} & sin\ ntT^{\frac{1}{2}} \\ -sin\ ntT^{\frac{1}{2}} & cos\ ntT^{\frac{1}{2}} \end{pmatrix} = U(nt),$$

$||U^n(t)|| = 1, n \in \mathbb{N}$ (nonexpanding group); $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T$, $\alpha = (\alpha_1, \alpha_2)^T$, $f(t) = (f_1(t), f_2(t))^T$. Solutions of equation (1) can be represented in the next form

$$\varphi(t) = U(t)c + \int_0^t U(t)U^{-1}(\tau)f(\tau)d\tau,$$

for any element $c \in H_T$. Substitute in condition (2) we obtain that solvability of boundary value problem (1), (2) is equivalent solvability the next operator equation

$$(I - U(w))c = g, \tag{3}$$

where $g = \alpha + U(w)\int_0^w U^{-1}(\tau)f(\tau)d\tau$. Consider the case when the set of values of $I - U(w)$ is closed $R(I - U(w)) = \overline{R(I - U(w))}$. Since $||U^n(w)|| = ||U(wn)|| = 1$ for all $n \in \mathbb{N}$ then [2] the operator system (3) is solvable if and only if

$$U_0(w)g = 0, \tag{4}$$

where

$$U_0(w) = \lim_{n\to\infty} \frac{\sum_{k=0}^n U^k(w)}{n} = \lim_{n\to\infty} \frac{\sum_{k=0}^n U(kw)}{n} -$$

orthoprojector, which projects the space H_T onto subspace $1 \in \sigma(U(w))$. Under this condition solutions of (3) have the form

$$c = U_0(w)\overline{c} + (\sum_{k=0}^{\infty}(\mu - 1)^k\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(w) - U_0(w))^l\}^{k+1} - U_0(w))g,$$

for $\mu > 1, |1 - \mu| < \frac{1}{||R_\mu(U(w))||}$ and any $\overline{c} \in H_T$. Then we can formulate first result as lemma.

Lemma 1. *Let the operator $I - U(w)$ has a closed image $R(I - U(w)) = \overline{R(I - U(w))}$.*

1. There exist solutions of boundary value problem (1), (2) if and only if

$$U_0(w)(\alpha + \int_0^w U^{-1}(\tau)f(\tau)d\tau) = 0. \tag{5}$$

2. Under condition (5), solutions of (1), (2) have the form

$$\varphi(t, \overline{c}) = U(t)U_0(w)\overline{c} + (G[f, \alpha])(t), \tag{6}$$

where

$$(G[f, \alpha])(t) = U(t)\sum_{k=0}^{\infty}(\mu - 1)^k\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(w) - U_0(w))^l\}^{k+1}(\alpha + \int_0^w U(w)U^{-1}(\tau)f(\tau)d\tau) -$$

$$-U(t)U_0(w)(\alpha+\int_0^w U(w)U^{-1}(\tau)f(\tau)d\tau)+\int_0^t U(t)U^{-1}(\tau)f(\tau)d\tau,$$

is the generalized Green operator of the boundary value problem (1), (2) for $\mu > 1, |1-\mu| < 1/||R_\mu(U(w))||$.

Now we show that condition $R(I-U(w)) = \overline{R(I-U(w))}$ of lemma 1 can be omitted and in the different senses boundary value problem (1), (2) is always resolvable.

1) Classical generalized solutions.

Consider case when the set of values of $I-U(w)$ is closed ($R(I-U(w)) = \overline{R(I-U(w))}$). Then [3] $g \in R(I-U(w))$ if and only if $\mathcal{P}_{N(I-U(w))^*)}g = 0$ and the set of solutions of (3) has the form [3] $c = G[g] + U_0(w)\overline{c}, \forall \overline{c} \in H_T$, where [2] and [3]

$$G[g] = (I-U(w))^+g = ((I-(U(w)-U_0(w))^{-1} - U_0(w))g$$

is generalized Green operator (or in the form of convergent series).

2) Strong generalized solutions. Consider the case when $R(I-U(w)) \neq \overline{R(I-U(w))}$ and $g \in \overline{R(I-U(w))}$. We show that operator $I-U(w)$ may be extended to $\overline{I-U(w)}$ in such way that $R(\overline{I-U(w)})$ is closed.

Since the operator $I-U(w)$ is bounded the next representation of H_T in the direct sum is true

$$H_T = N(I-U(w)) \oplus X, H_T = \overline{R(I-U(w))} \oplus Y,$$

with $X = N(I-U(w))^\perp = \overline{R(I-U(w))}$ and $Y = \overline{R(I-U(w))}^\perp = N(I-U(w))$. Let $E = H_T/N(I-U(w))$ is quotient space of H_T , $\mathcal{P}_{\overline{R(I-U(w))}}$ and $\mathcal{P}_{N(I-U(w))}$ are orthoprojectors, which project onto $\overline{R(I-U(w))}$ and $N(I-U(w))$ respectively. Then operator

$$\mathcal{I}-\mathcal{U}(w) = \mathcal{P}_{\overline{R(I-U(w))}}(I-U(w))j^{-1}p : X \to R(I-U(w)) \subset \overline{R(I-U(w))},$$

is linear, continuous and injective. Here

$$p : X \to E = H_T/N(I-U(w)), \quad j : H_T \to E$$

are continuous bijection and projection respectively. The triple (H_T, E, j) is a locally trivial bundle with typical fiber $H_1 = \mathcal{P}_{N(I-U(w))}H$ [4]. In this case [5, p.26,29] we can define strong generalized solution of equation

$$(\mathcal{I}-\mathcal{U}(w))x = g, x \in X. \tag{7}$$

Fill up the space X in the norm $||x||_{\overline{X}} = ||(\mathcal{I}-\mathcal{U}(w))x||_F$, where $F = \overline{R(I-U(w))}$ [5]. Then extended operator

$$\overline{\mathcal{I}-\mathcal{U}(w)} : \overline{X} \to \overline{R(I-U(w))}, X \subset \overline{X}$$

is homeomorphism of $\overline{X}$ and $\overline{R(I-U(w))}$. By virtue of construction of strong generalized solution [5] equation

$$(\overline{\mathcal{I}-\mathcal{U}(w)})\overline{\xi} = g,$$

has a unique solution $(\overline{\mathcal{I}-\mathcal{U}(w)})^{-1}g$ which is called generalized solution of equation (7).

Remark 1. It should be noted that there are exists next extensions of spaces and corresponding operators

$$\overline{p} : \overline{X} \to \overline{E}, \ \ \overline{j} : \overline{H}_T \to \overline{E}, \ \ \overline{\mathcal{P}_X} = \mathcal{P}_{\overline{X}} : \overline{H}_T \to \overline{X}, \ \ \overline{G} : \overline{R(I-U(w))} \to \overline{X},$$

where

$$\overline{H}_T = N(I-U(w)) \oplus \overline{X}; \ \ \overline{p}(x) = p(x), x \in X; \ \ \overline{j}(x) = j(x), x \in H_T,$$

$$\overline{\mathcal{P}}_X(x) = \mathcal{P}_X(x), x \in H_T \ (\mathcal{P}_X = \mathcal{P}_X^2 = \mathcal{P}_X^*); \ \ \overline{G}[g] = G[g], g \in R(I-U(w)).$$

Then the operator $\overline{I-U(w)} = (\overline{\mathcal{I}-\mathcal{U}(w)})\mathcal{P}_{\overline{X}} : \overline{H}_T \to \overline{H}_T$ is extension of $I-U(w)$, $\overline{(I-U(w))}c = (I-U(w))c$ for all $c \in H_T$.

3) Strong pseudosolutions.

Consider element $g \notin \overline{R(I-U(w))}$. This condition is equivalent $\mathcal{P}_{N(I-U(w))^*)}g \neq 0$. In this case there are exists elements from $\overline{H_T}$ which minimise norm $||\overline{(I-U(w))}\xi - g||_{\overline{H_T}}$:

$$\xi = \overline{(\mathcal{I}-\mathcal{U}(w))}^{-1}g + \mathcal{P}_{N(I-U(w))}\overline{c}, \forall \overline{c} \in H_T.$$

These elements we call *strong pseudosolutions* by analogy of [3].

Now we formulate the full theorem of solvability.

Theorem 1. *Boundary value problem (1), (2) is always resolvable.*

1) a) There are exists classical or strong generalized solutions of (1), (2) if and only if

$$U_0(w)(\alpha + \int_0^w U^{-1}(\tau)f(\tau)d\tau) = 0. \tag{8}$$

If $(\alpha + \int_0^w U^{-1}(\tau)f(\tau)d\tau) \in R(I-U(w))$ *then solutions of (1), (2) will be classical.*

b) Under assumption (8) solutions of (1), (2) have the form

$$\varphi(t, \overline{c}) = U(t)U_0(w)\overline{c} + (\overline{G[f, \alpha]})(t),$$

where $(\overline{G[f,\alpha]})(t)$ *- is extension of operator* $(G[f,\alpha])(t)$*;*

3) a) There are exists strong pseudosolutions if and only if

$$U_0(w)(\alpha + \int_0^w U^{-1}(\tau)f(\tau)d\tau) \neq 0. \tag{9}$$

b) Under assumption (9) strong pseudosolutions of (1), (2) have the form

$$\varphi(t,\overline{c}) = U(t)U_0(w)\overline{c} + (\overline{G[f,\alpha]})(t),$$

where

$$(\overline{G[f,\alpha]})(t) = U(t)\overline{G}[g] + \int_0^t U(t)U^{-1}(\tau)f(\tau)d\tau = U(t)\overline{(\mathcal{I}-\mathcal{U}(w))}^{-1}g + \int_0^t U(t)U^{-1}(\tau)f(\tau)d\tau.$$

Main result (Nonlinear case). Generalization of Lyapunov-Schmidt method

In the Hilbert space H_T defined below we consider the boundary value problem

$$\frac{d\varphi(t)}{dt} = -iH_0\varphi(t) + \varepsilon Z(\varphi(t),t,\varepsilon) + f(t), \tag{10}$$

$$\varphi(0,\varepsilon) - \varphi(w,\varepsilon) = \alpha. \tag{11}$$

We seek a bounded solution $\varphi(t,\varepsilon)$ of boundary value problem (10), (11) that becomes one of the solutions of the generating equation (1), (2) $\varphi_0(t,\overline{c})$ in the form (6) for $\varepsilon = 0$.

To find a necessary condition on the operator function $Z(\varphi,t,\varepsilon)$, we impose the joint constraints

$$Z(\cdot,\cdot,\cdot) \in C([0;w],H_T) \times C[0,\varepsilon_0] \times C[||\varphi - \varphi_0|| \leq q],$$

where q is some positive constant.

The main idea of the next results is presented in [6] for investigating of bounded solutions.

Let us show that this problem can be solved with the use of the operator equation for generating amplitudes

$$F(\overline{c}) = U_0(w)\int_0^w U^{-1}(\tau)Z(\varphi_0(\tau,\overline{c}),\tau,0)d\tau = 0. \tag{12}$$

Theorem 2. (necessary condition) *Let the nonlinear boundary value problem (10), (11) has a bounded solution $\varphi(\cdot,\varepsilon)$ that becomes one of the solutions $\varphi_0(t,\overline{c})$ of the generating equation (1), (2) with constant $\overline{c} = c^0$, $\varphi(t,0) = \varphi_0(t,c^0)$ for $\varepsilon = 0$. Then this constant should satisfy the equation for generating amplitudes (12).*

To find a sufficient condition for the existence of solutions of boundary value problem (10), (11) we additionally assume that the operator function $Z(\varphi,t,\varepsilon)$ is strongly differentiable in a neighborhood of the generating solution ($Z(\cdot,t,\varepsilon) \in C^1[||\varphi-\varphi_0|| \le q]$).

This problem can be solved with the use of the operator

$$B_0 = \frac{dF(\overline{c})}{d\overline{c}}|_{\overline{c}=c_0} = \int_{-\infty}^{+\infty} H(t)A_1(t)T(t,0)P_+(0)\mathcal{P}_{N(D)}dt : H \to H,$$

where $A_1(t) = Z^1(v,t,\varepsilon)|_{v=\varphi_0,\varepsilon=0}$ (the Fréchet derivative).

Theorem 3. (sufficient condition) *Let the operator B_0 satisfy the following conditions:*

1) The operator B_0 is Moore-Penrouse pseudoinvertible;

2) $\mathcal{P}_{N(B_0^*)}U(w) = 0$.

Then for arbitrary element $c = c^0 \in H_T$, satisfying the equation for generating amplitudes (12), there exists at least one solution of (10), (11).

This solution can be found with the use of the iterative process:

$$\overline{v}_{k+1}(t,\varepsilon) = \varepsilon G[Z(\varphi_0(\tau,c^0) + v_k,\tau,\varepsilon),\alpha](t),$$

$$c_k = -B_0^+U_0(w)\int_0^w U^{-1}(\tau)\{A_1(\tau)\overline{v}_k(\tau,\varepsilon) + \mathcal{R}(v_k(\tau,\varepsilon),\tau,\varepsilon)\}d\tau,$$

$$v_{k+1}(t,\varepsilon) = U(t)U_0(w)c_k + \overline{v}_{k+1}(t,\varepsilon),$$

$$\varphi_k(t,\varepsilon) = \varphi_0(t,c^0) + v_k(t,\varepsilon), k = 0,1,2,..., \quad v_0(t,\varepsilon) = 0, \varphi(t,\varepsilon) = \lim_{k\to\infty}\varphi_k(t,\varepsilon).$$

Remark 2. Proof of theorems 2 and 3 follows directly from works [6], [7].

Relationship between necessary and sufficient conditions.

First, we formulate the following assertion.

Corollary. *Let a functional $F(\overline{c})$ have the Fréchet derivative $F^{(1)}(\overline{c})$ for each element c^0 of the Hilbert space H satisfying the equation for generating constants (12). If $F^1(\overline{c})$ has a bounded inverse, then boundary value problem (10), (11) has a unique solution for each c^0.*

Remark 3. If assumptions of the corollary are satisfied, then it follows from its proof that the operators B_0 and $F^{(1)}(c^0)$ are equal. Since the operator $F^{(1)}(\overline{c})$ is invertible, it

follows that assumptions 1 and 2 of Theorem 3 are necessarily satisfied for the operator B_0. In this case, boundary value problem (10), (11) has a unique bounded solution for each $c^0 \in H_T$ satisfying (12). Therefore, the invertibility condition for the operator $F^1(\overline{c})$ relates the necessary and sufficient conditions. In the finite-dimensional case, the condition of invertibility of the operator $F^{(1)}(\overline{c})$ is equivalent to the condition of simplicity of the root c^0 of the equation for generating amplitudes [3].

In such way we generalize the well-known method of Lyapunov-Schmidt. It should be emphasized that theorem 2 and 3 give us condition of chaotic behavior of (10), (11) [8].

Example. Now we illustrate obtained assertion. Consider the next differential equation in separable Hilbert space H

$$\ddot{y}(t) + Ty(t) = \varepsilon(1 - ||y(t)||^2)\dot{y}(t), \tag{13}$$

$$y(0) = y(w), \quad \dot{y}(0) = \dot{y}(w), \tag{14}$$

where T is unbounded operator with compact T^{-1}. Then there is exists orthonormal basis $e_i \in H$ such that $y(t) = \sum_{i=1}^{\infty} c_i(t)e_i$ and $Ty(t) = \sum_{i=1}^{\infty} \lambda_i c_i(t)e_i$, $\lambda_i \to \infty$. Operator system (10), (11) for boundary value problem (13), (14) in this case will be equivalent the next countable system of ordinary differential equations ($c_k(t) = x_k(t)$)

$$\dot{x}_k(t) = \sqrt{\lambda_k} y_k(t), \quad k = 1, 2, ...,$$

$$\dot{y}_k(t) = -\sqrt{\lambda_k} x_k(t) + \varepsilon\sqrt{\lambda_k}(1 - \sum_{j=1}^{\infty} x_j^2(t))y_k(t), \tag{15}$$

$$x_k(0) = x_k(w), y_k(0) = y_k(w). \tag{16}$$

We find solutions of these equations such that for $\varepsilon = 0$ turns in one of the solutions of generating equation. Consider critical case $\lambda_i = 4\pi^2 i^2/w^2, i \in \mathbb{N}$. Let $w = 2\pi$. In that case the set of all periodic solutions of (15), (16) have the form

$$x_k(t) = cos(kt)c_1^k + sin(kt)c_2^k,$$

$$y_k(t) = -sin(kt)c_1^k + cos(kt)c_2^k,$$

for all pairs of constant $c_1^k, c_2^k \in \mathbb{R}, k \in \mathbb{N}$. Equation for generating amplitudes (12) in this case will be equivalent the next countable systems of algebraic nonlinear equations

$$(c_1^k)^3 + 2 \sum_{j=1, j\neq k} (c_1^k(c_1^j)^2 + c_1^k(c_2^j)^2) + c_1^k(c_2^k)^2 - 4c_1^k = 0,$$

$$(c_2^k)^3 + 2 \sum_{j=1, j\neq k} (c_2^k(c_1^j)^2 + c_2^k(c_2^j)^2) + (c_1^k)^2 c_2^k - 4c_2^k = 0, k \in \mathbb{N}.$$

Then we can obtain the next result

Theorem 4(necessary condition of van der Pol's equation).*Let the boundary value (15), (16) have a bounded solution $\varphi(\cdot, \varepsilon)$ that becomes one of the solutions of the generating equations with pairs of constant $(c_1^k, c_2^k), k \in \mathbb{N}$. Then only finite number of these pairs are not equal zero. Moreover, if $(c_1^{k_i}, c_2^{k_i}) \neq (0,0), i = \overline{1, N}$ then these constants lie on N-dimensional torus in infinite dimensional space of constants*

$$(c_1^{k_i})^2 + (c_2^{k_i})^2 = (\frac{2}{\sqrt{2N-1}})^2, i = \overline{1, N}.$$

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